Memorandum For A Reasonable Sequence

October 28th, 2008

Having ended the previous entry in these **NOTES** with an announcement that I would "discuss developmental mathematics as embodied in the Arithmetic-Basic Algebra-Differential Calculus sequence", I found myself somewhat at a loss as to exactly how to resume matters.

Fortunately, being known at my school as a militant, radical fault-finder of **Developmental Mathematics** courses as currently practiced, and therefore a royal pain, I was recently asked by a presumably sick and tired administrator how I would specify a program intended for the kind of students currently enrolling into **Developmental Algebra**. Obliging as ever in such cases, I wrote a few pages to sketch a program that would bring these students, in three four-credit semesters and with an acceptable success rate, to the level achieved in **Calculus I** (*Differential*) and to explain what it is that would make such a program work.

What will ensue at my school is anyone's guess but it seemed to me that I might as well publish here (a slightly edited version of) that sketch as it might be of some small use to someone out there.

NOTE: So far, I have not been able to convert the formulas written below in LaTeX into HTML but I am still trying. In the meantime, I apologize.

Description Of The Sequence

The most important part of such a program is of course the *contents* of the three courses in the sequence and, most especially, their *architecture*. One possibility would be for the sequence to consist of:

1. An **Arithmetic-Algebra** course, to be discussed at some length below, specially and specifically designed to serve as an introduction to:
2. An **Algebraic Functions** course dealing with their introduction, algebraic discussion and *differential calculus,*
3. A **Transcendental Functions** course dealing with their introduction, algebraic discussion and *differential calculus*.

Of course, one reason for this *architecture* is that it is that of the two courses, **Differential Calculus I** and **Differential Calculus II**, that were originally specified, designed and experimented with under the terms of a 1988 **NSF** grant as an alternative to the conventional sequence, **Precalculus I**, **Precalculus II** and **Calculus I** (*Differential*). The 1992 report of the school's Office of Institutional Research said in part:

"Of those attempting the first course in each sequence, 12.5% finished the [conventional three semester 10 hour] sequence while 48.3% finished the [integrated two semester 8-hour] sequence, revealing a definite association between the [integrated two semester 8 hour] sequence and completion ($\chi 2$ (1) = 82.14, $p < .001$).”

The report also said that the passing rates in **Calculus II** (*Integral*) for the students coming from the above two sequences were almost identical but that this was not significant because most students did not continue into **Calculus II** (*Integral*).

However, for reasons that may or may not be obscure, the integrated sequence all but died out. (This is of course what happened to most, if not all, the courses and textbooks developed under the NSF grant program. In particular, this was the eventual fate of the book developed at Harvard under, if memory serves, a one and a quarter million dollars NSF grant.)

The approach used in **Differential Calculus I** and **Differential Calculus II** to bypass a lot of the usual stumbling blocks on the and to provide the students with a continuous and sustained conceptual development is due to Lagrange, one of the preeminent nineteenth century mathematicians, who wanted to avoid the use of *limits* in the development of calculus.
While the standard objection to Lagrange’s treatment is that it cannot deal with *all* functions, it certainly does handle all functions likely to be encountered by students in **Calculus I** (*Differential*) and can serve as a basis from which to develop the Bolzano-Cauchy-Weirstrass treatment of limits that is nowadays de rigueur in, or at least underlies, “elementary” calculus texts. What it cannot deal with is only the kind of esoteric functions encountered by research mathematicians.
The details of how Lagrange’s approach achieves this is beyond the scope of this paper but an important fact to keep in mind in terms of the sequence sketched here is that it is based solely on the use of something called *polynomial approximations*. (More about this below.)

Description Of The Arithmetic-Algebra Course

Because of the work done for **Differential Calculus I** and **Differential Calculus II**, the contents of the **Algebraic Functions** course and of the **Transcendental Functions** course are already well specified and therefore the contents of the **Arithmetic-Algebra** course are fairly well specified as being exactly, no more, no less, what is needed for the **Algebraic Functions** and **Transcendental Functions** courses.
In fact, the package ***Reasonable Basic Algebra*** already contains a large part of what is needed, namely

- Elementary equations and “inequations”,
- Laurent polynomials i.e. polynomials that can include *negative* powers.

What is not clear is exactly how much to do with these contents to ensure a “profound understanding”—in the sense of Liping Ma’s “*profound understanding of fundamental mathematics*“. So, here, some experimentation would certainly be necessary.

In any case, what is missing from the ***Reasonable Basic Algebra*** package, yet is absolutely necessary for a profound understanding of **Algebraic Functions**and **Transcendental Functions**, is a profound understanding of three more concepts:

- Decimal Numbers

- Functions
- Approximations

While the first part of the ***Reasonable Basic Algebra*** package does deal with numbers, it does so mostly with *counting* numbers, *plain* and *signed*. But, even though *decimal* numbers are constantly used in Part 2 and 3, for lack of time, the ***Reasonable Basic Algebra*** package could not discuss *decimal numbers* per se and had to take for granted their profound understanding. This of course precluded any discussion of *Approximations*.

Similarly, for lack of time, *functions* are not really introduced in the ***Reasonable Basic Algebra*** package even though, without saying it, operations *are* introduced as *functions*, that is, for instance, as adding 2 *to* 3 which can be visualized as

$latex 3 \xrightarrow{\hspace{2mm} +2\hspace{2mm}}\;5$

as opposed to adding 3 *and* 2 which can be visualized as

$latex 3, 2 \xrightarrow{\hspace{2mm} +\hspace{2mm}}\;5$

The distinction may appear overly subtle but, in fact, it facilitates considerably the understanding of how we work with numbers. For instance,

- We can then say that subtracting 2 "undoes" adding 2; visually:
  $LATEX 3 \xrightarrow{\hspace{2mm} +2\hspace{2mm}}5
  \xrightarrow{\hspace{2mm} -2\hspace{2mm}}3$

- Similarly, the equation $x + 2 = 5$ can be understood as the question "to what number should we add 2 to get 5?". Visually:
  $latex x \xrightarrow{\hspace{2mm} +2\hspace{2mm}}5$

Had *functions* been actually introduced, the above question would then have been recast as:

For which input(s), if any, does the function whose input-output rule is

$latex x \xrightarrow{\hspace{2mm} JILL\hspace{2mm}}JILL(x) = x+2$

return the output 5?

The latter is of course cumbersome but it does familiarize the students with a way to look at things that is crucial in **Algebraic Functions** where we might ask, for instance,

For which input(s), if any, does the function whose input-output rule is

$latex x\xrightarrow{\hspace{2mm}QUAD\hspace{2mm}}QUAD(x) = -3x^{2}+5x-7$

return the output -23?

which leads naturally to the *quadratic equation* [pmath] -3x^2+5x-7= -23[/pmath]

which can then be solved simply by counting inputs from the *vertex*.

Thus, the **Arithmetic-Algebra** course should be designed something as follows:

1. Introduce *functions* as the mathematical version of real-world input-output devices.
2. Introduce *counting numbers*, *plain* and *signed*, essentially along the lines of the ***Reasonable Basic Algebra*** package but now in the context, and with the full aid, of *functions*. In particular, plotting functions when the *data set* consists of signed counting numbers would go a long way towards the students' familiarization with these.
3. Introduce *decimal numbers* with, immediately, the then necessary concept of *approximation*. For instance, while the question
   For which input(s), if any, does the function whose input-output rule is
   $latex x \xrightarrow{\hspace{2mm} DIL\hspace{2mm}}DIL(x) = 3x$

   return the output 12?

   has the solution 4, the question
   For which input(s), if any, does the function whose input-output rule is
   $latex x \xrightarrow{\hspace{2mm} DIL\hspace{2mm}}DIL(x) = 3x$

   return the output 12?

   does not admit of any *exact* decimal solution. However, we may say that any of the following
   4 + [...]
   4.3 + [...]
   4.33 + [...]
   4.333 + [...]

   where [...] is to be read as "something too small to matter in the current situation" is a solution.
   Which *approximation* we will choose will depend of course on the actual situation and so, in the meantime, we will say that the solution is
   $latex \dfrac{13}{4}$

   which is then read as "code" for setting up the division of 4 into 13 with the decision as to where to stop the division left to when we know what the actual situation requires.
4. Very little in the investigation of *equations* and *inequations* in Part 2 of the ***Reasonable Basic Algebra*** package would have to be modified to take full advantage of the above.
5. In fact, and more generally, once the sequence is a given, there would be no need to keep the contents in the particular courses described above and, presumably, one could vastly improve the learning curve by bringing in the contents on an "as needed" basis.It is for instance quite probable that the study of *affine functions* currently in the **Algebraic Functions** course could occur in the **Arithmetic-Algebra** course where

affine equations are already dealt while leaving most of the study of *approximations* for the investigation of *quadratic function*s and *cubic functions* in the **Algebraic Functions** course where it is really needed. (Of course, the courses would then have to be renamed accordingly.)

Rationale

Some explanation as to why the contents have to be something like what was described above is probably in order.

1. The students entering **Developmental Mathematics** courses are "damaged" in the sense that they want only to be shown "how to do" the problems understood to appear on the exam. When reminded that "show and tell, drill and test" is precisely what got them into **Developmental Mathematics** courses, many even rise to the defense of their school/teachers and say that this is only because "they were never good in math".
2. Compounding the problem is the fact that all commercially available textbooks are memory based and that, to facilitate memorization, the subject matter is atomized into "topics" that are then presented independently of each other. All connective tissues have been removed and, as a result, nothing can make sense anymore and no build-up can take place. Indeed, typically, instructors deplore that the students cannot remember the simplest things past the test.
3. Given these circumstances, such a "fast track" as described above would seem utterly improbable. The point, though, is that the content architecture sketched above is extremely efficient in that everything serves to support all that follows and thus fosters an environment in which the students are able and begin to see that things mathematical are the way they are, not because someone or some book says so, but because of the way they connect to each other so that it makes sense that they should be the way they are.
   For example, consider the number 2345.67. In the **Arithmetic-Algebra** course, it initially stands as a shorthand for
   $latex 2\cdot10^{+3} + 3\cdot10^{+2} + 4\cdot10^{+1} + 5\cdot10^{0} + 6\cdot10^{-1} + 7\cdot10^{-2}$

   to represent
   $latex 2 Clevelands \& 3 Franklins \& 4 Hamiltons \& 5 Wahingtons \& 6 Dimes \& 7 Cents$

   The exponents are "code" for the number of 0s to be placed *after* the 1 when the exponent is + or to be placed *before* the 1 when the exponent is -:
   $latex \begin{align*}
   2345.67
   &=2\cdot1000. + 3\cdot100. + 4\cdot10. + 5\cdot1. + 6\cdot0.1 + 7\cdot0.01
   \\
   &= 2000. + 300. + 40. + 5. + 0.6 + 0.07
   \end{align*} $
   With the details that are quite necessary but omitted here, the above makes complete sense to the students. Most important is that the concept of multiplication is not required

at that point and that this is all that is needed to start the development of arithmetic (In the above, the symbol + should be read as “and”).
For instance, the above is enough to discuss why, depending on the situation, we can write
$latex \begin{align*}
2345.67
&= 2000 + [...]
\\
&= 2300 + [...]
\\
&= 2340 + [...]
\\
&= 2345 + [...]
\\
&= 2345.6 + [...]
\end{align*}
$
But then, once we have introduced multiplication, we realize that, say,

$latex 2 · 10^{+3}$

can be read as

2 *multiplied* by 3 copies of 10

and

$latex 2 · 10^{-3}$

can be read as

2 *divided* by 3 copies of 10

From there, it is an easy transition to reading
$latex 2 · x^{+3}$

as
2 *multiplied* by 3 copies of x

and reading
$latex 2 · x^{-3}$

as
2 *divided* by 3 copies of x

And then, we can see why, when x stands for a *large number*, we can write, depending on the situation,

$$\begin{align*}
\hspace{-7mm}2x^{+3}+3x^{+2}+4x^{+1}+5x^{0}+6x^{-1}+7x^{-2}
&=2x^{3} + [...]
\\
&=2x^{+3}+3x^{+2} + [...]
\\
&=2x^{+3}+3x^{+2} +4x^{+1} +[...]
\\
&=2x^{+3}+3x^{+2} +4x^{+1}+5x^{0} +[...]
\\
&=2x^{+3}+3x^{+2} +4x^{+1}+5x^{0}+6x^{-1} +[...]
\end{align*}$$

A profound understanding of *decimal numbers* and of how they are *approximated* is absolutely crucial to the profound understanding of functions afforded by Lagrange's viewpoint because not only are *polynomial functions* approximated essentially in the same manner as decimal numbers but *polynomial functions* in fact serve to approximate *all* the functions normally encountered in calculus in exactly the same manner that *decimal numbers* serve to approximate *all* numbers, for instance $\sqrt{2}$, π, e, the golden ratio, etc.

4. Eventually, as the students's understanding deepens, the attitude mentioned above begins to change and many students begin to be more willing to take the time to consider a question, what it means and how to cope with it, one way or the other, and then to take whatever more time it takes to get a result and/or to make a case for whatever result they have come to.
   Incidentally, the amount of time students are willing to stay on a given question is an extremely good indicator of their progress in the direction of thinking for themselves. Significantly, the questions to the instructor get to be less and less about whether they "got it wrong" and more and more about where they made the wrong turn.
5. However, the problem is that this change in attitude starts being noticeable only about two thirds of the way down the *first* course. And of course, if the second course is "standard", everything goes back to square one: "teach me, show me". In other words, the convalescence cannot be expected to take place within a single semester.
6. Some students, though, have already reached a "denial" stage where, against all evidence, continue to say that they will just memorize and pass the course.

Practical Considerations

While a content architecture such as the one sketched above is *necessary*, it is unfortunately not *sufficient* to ensure acceptable retention and success rates.

1. Students need to be able to *read* (and write) mathematical explanations and this is far from being initially the case. It is thus necessary to link the sections of the **Arithmetic-Algebra** course with sections of a **Developmental English Reading** course in which

the texts assigned for reading are those used in the **Arithmetic- Algebra** course: Textbook and Review DISCUSSIONS.
The link was tried once in my school and the instructor who had taught the **Remedial English Reading** course later wrote that "*The students that stayed to the end also appreciated* [*the approach*] *whether they passed or not. If we pursue another link, the English teacher should definitely read the math text with the students. Unfortunately, because I had my own reading to do, we did not read the math in English class as we should have done*."

2. Students need to have the *time* necessary to discover and experience what is for them a completely new modus operandi, namely "thinking" as opposed to memorizing.
There should therefore be a "study period", after each class, in a manner of a lab, to ensure that the students *will have* the necessary time to do the homeworks provided in the package while reading the Textbook and the Review DISCUSSION. (See for example the ***Reasonable Basic Algebra*** package). Just as with physics or chemistry labs, an instructor should be present as a "resource".
3. There should be as much *continuity* as possible:
    1. The **Arithmetic-Algebra** course should be offered only in the Fall with the **Algebraic Functions** course offered in the Spring and the **Transcendental Function**s course offered in the following Fall (or in the Summer but only in a fourteen week course.)
    2. The instructor ought to be able to follow her/his students from one course to the next throughout the entire sequence—except of course when students need to repeat a class.
    3. Some "contract" should be passed between entering students and the school to ensure:
        - That the students will attend classes and study periods,
        - That the school will offer the subsequent courses on schedule,
        - That the school will give the students a specific number of credits upon completion of the sequence,
        - That the school will have seen to it that the sequence transfers appropriately in four-year schools.
    4. Some screening would seem to be necessary but should absolutely not be done on the basis of "knowledge" and *only* to determine the likelihood that the student's commitment is realistic and seriously understood.
    There is a subtle difference between students testing into **Developmental Arithmetic** and students testing into **Developmental Algebra**, although not entirely in favor of the **Developmental Algebra** students. So, although it seems a priori safer to start such a program with students testing into **Developmental Algebra**, once a screening has been found to successfully predict the students' level of commitment, there does not remain much reason for not accepting in the program students testing into **Developmental Arithmetic** so that, eventually, the distinction could be dispensed with entirely.

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